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The use of stokeslets to describe the arbitrary translation of a disk near a plane wall

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Abstract. A previously presented method is extended to describe the fully three-dimensional Stokes flow generated by the translation in any direction of an arbitrarily oriented disk in fluid bounded by a plane wall. The velocity field is represented solely in terms of stokeslet distributions on the disk, modified to take account of the bounding wall according to the century-old idea of Lorentz. Sets of integral equations of the second kind, not all disjoint, are obtained for the Abel transforms in each Fourier mode of the density functions. However, only a few modes need be considered in determining the flow field to order D^{-3}, where D is the distance of the disk axis from the wall. Less detail is required to evaluate the drag force and torque experienced by the disk.

1. Introduction

The slow rate of decay of creeping flow disturbances makes it necessary to take proper account of rigid boundaries in order to give an accurate description of the motion. This enhances the difficulty of the calculation, particularly when a loss of symmetry thus occurs, and if a fully three-dimensional flow is to be considered there is little chance of finding any type of analytical solution. Many authors [1–8] have presented solutions for flows that combine transverse motion with axisymmetric geometry, since in these cases only the first Fourier mode is excited, so the extra difficulty is limited to a sixth-order instead of a fourth-order problem. Recently, the present author [9] considered a disk sedimenting towards a plane wall, a flow problem which retains some symmetry but is fully three-dimensional in requiring all Fourier modes to be included. Attempts to establish a solvable, infinite set of dual integral equations, as for the disk in isolation [7] or the shear flow past a hole in a plane [10,11], were unsuccessful, so the more basic method of using tangentially directed stokeslets, modified to take account of the wall, was adopted. The introduction of Abel transforms in each mode then enabled the complete solution to be formally obtained and, finally, a suitable truncation of the system, leading to simple polynomials for the Abel transforms of the modal density functions, was obtained by expanding the reflected velocities in inverse powers of the distance D of the disk's center from the wall. In this way, the dimensionless drag coefficient was evaluated to order D^{-5}.

In reality, a solution by means of sets of dual integral equations is often precluded by the presence of fixed boundaries. The method has been successfully used [8] for the disk moving edgewise parallel to a wall or free surface, the influence of a finite boundary on the shear flow disturbance due to a circular hole in the fixed bounding plane, and the edgewise oscillations of a disk in unbounded fluid; all of these flow problems can be solved by essentially the same procedure. More often, though, the successful solution can be identified as equivalent to the use of stokeslet singularities, modified to take account of the bounding walls in the manner suggested by Lorentz [12]. Obvious examples in which this fact is not stated are [13], which has a disk inside a cylinder, and [14], which starts with a disk near a plane wall or free surface, both problems being totally axisymmetric. In the latter, the stream function

$$\frac{2U}{\pi}\int_0^\infty \rho J_1(k\rho)e^{-k|z|}[A(k)(k^{-1}+|z|)+B(k)z]\,\mathrm{d}k$$

can be identified as that due to distributions over the disk ($0 \leqslant \rho \leqslant 1, z = 0$ in cylindrical polar coordinates) of normally and radially directed stokeslets, because it is readily shown that a ring of force singularities of strength $-\hat{\mathbf{z}}$ at $\rho = \rho'$ generates the stream function

$$\frac{1}{4\mu}\int_0^\infty \rho J_1(k\rho)e^{-k|z|}(k^{-1}+|z|)J_0(k\rho')\,\mathrm{d}k,$$

while a ring of force singularities of strength $-\hat{\rho}$ at $\rho = \rho'$ generates the stream function

$$\frac{1}{4\mu}\int_0^\infty \rho J_1(k\rho)e^{-k|z|}zJ_1(k\rho')\,\mathrm{d}k.$$

In this paper, a fully three-dimensional flow problem is solved by exploiting the advantage that the disturbance generated by the arbitrary translation of a disk can be completely represented by a distribution of variously oriented stokeslets over the disk. The calculation is a more elaborate version of that given in [9]. Stokeslets in all three directions are now required and each density function has both even and odd Fourier components. The alignment of these singularities is normal and tangential to the disk, since there is a definite distinction between the treatment of the broadside and edgewise components of the disk's motion. This requires appropriate resolution of the reflected velocities, which are expanded in powers of D^{-1} in order once again to achieve a neat truncation of the system of integral equations of the second kind. The $O(D^{-1})$ solution involves only constant components of the reflected velocities and yields results that, when the disk is aligned parallel or perpendicular to the wall, reduce to those given by Brenner [15]. The Abel transforms of the nth mode density functions have a leading term of type s^n/D^{n+1} due to the presence of the wall. Thus, for an $O(D^{-3})$ solution, modes beyond the second can be discarded and it turns out that the drag force can be evaluated without determination of the second mode functions, while the first mode functions yield the torque, of order D^{-2}, experienced by the disk.

2. Formulation of the disk problem

A thin rigid disk of unit radius translates steadily in an incompressible viscous fluid that is at rest at infinity and bounded by a wall at distance D from the centre of the disk. Cartesian coordinates (x, y, z) are chosen so that instantaneously the centre of the disk is at the origin, the wall is at $x = D$ and the disk lies in the plane $x\cos\beta = y\sin\beta$, i.e. the disk has been rotated about the z-axis through an angle β, regarded as positive acute, from the axisymmetric orientation. The Reynolds number of the viscous incompressible flow is assumed to be sufficiently small for the velocity field $\mathbf{v}$ to satisfy the creeping flow (Stokes) equations

$$\mu\nabla^2\mathbf{v} = \nabla p, \qquad \nabla\cdot\mathbf{v} = 0, \tag{1}$$

where μ is the coefficient of viscosity and p the dynamic pressure. The boundary conditions to be applied are

$$\mathbf{v} = U_0\hat{\mathbf{x}} + V_0\hat{\mathbf{y}} + W_0\hat{\mathbf{z}} \quad \text{at the disk,} \tag{2}$$

where $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ denote unit vectors, and the no-slip condition at the plane wall

$$\mathbf{v} = \mathbf{0} \quad \text{at} \quad x = D. \tag{3}$$

The special cases are:

(i) $\beta = 0$ (axisymmetric geometry) in which $U_0 = 0$ yields edgewise motion parallel to the wall while $V_0 = 0 = W_0$ gives broadside motion towards the wall, and

(ii) $\beta = \pi/2$ in which $V_0 = 0 = W_0$ and $U_0 = 0 = V_0$, respectively, yield edgewise motion towards and parallel to the wall, while $U_0 = 0 = W_0$ gives broadside motion parallel to the wall.

To isolate the broadside component of the disk's motion, introduce new coordinates (ξ, η) by setting

$$\xi = x\cos\beta - y\sin\beta, \qquad \eta = x\sin\beta + y\cos\beta. \tag{4}$$

The fluid motion generated by the translating disk can be completely represented by a distribution of stokeslets over the disk with density

$$\frac{1}{\pi^2}[e(\alpha,\phi)\hat{\boldsymbol{\xi}} + f(\alpha,\phi)\hat{\boldsymbol{\eta}} + g(\alpha,\phi)\hat{\mathbf{z}}],$$

at position $\xi = 0, (\eta, z) = \alpha(\cos\phi, \sin\phi)(0 \leqslant \alpha \leqslant 1, -\pi < \phi \leqslant \pi)$ on the disk. These functions have Fourier expansions of the form

$$e(\alpha,\phi) = e_0(\alpha) + 2\sum_1^\infty [e_{nc}(\alpha)\cos n\phi + e_{ns}(\alpha)\sin n\phi], \tag{5}$$

which, with ϵ_n denoting Neumann's symbol and $e_{0c} = e_0$, can be abbreviated as

$$e(\alpha,\phi) = \sum_0^\infty \epsilon_n [e_{nc}(\alpha)\cos n\phi + e_{ns}(\alpha)\sin n\phi].$$

Fourier coefficients of $f(\alpha,\phi)$ and $g(\alpha,\phi)$ are defined similarly. Evidently, $\{e_{nc}, f_{nc}, g_{ns}\}$ are generated by U_0 and V_0 and $\{e_{ns}, f_{ns}, g_{nc}\}$ by W_0.

As in [9], it is convenient to introduce Abel transforms, defined typically by

$$E_{nc}(s) = \frac{2}{\pi}\int_s^1 \left(\frac{s}{\alpha}\right)^n \frac{e_{nc}(\alpha)\alpha}{(\alpha^2 - s^2)^{1/2}}\,\mathrm{d}\alpha \quad (n \geqslant 0),$$

i.e.

$$e_{nc}(\alpha) = -\alpha^{n-1}\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_0^1 \frac{E_{nc}(s)s^{1-n}}{(s^2 - \alpha^2)^{1/2}}\,\mathrm{d}s \quad (n \geqslant 0). \tag{6}$$

The drag force $\mathbf{F}_D$ and torque $\mathbf{T}_D$ exerted by the fluid on the disk are then given by

$$\begin{aligned}\mathbf{F}_D &= -16\mu\int_0^1 [e_0(\alpha)\hat{\boldsymbol{\xi}} + f_0(\alpha)\hat{\boldsymbol{\eta}} + g_0(\alpha)\hat{\mathbf{z}}]\alpha\,\mathrm{d}\alpha \\ &= -16\mu\int_0^1 [E_0(s)\hat{\boldsymbol{\xi}} + F_0(s)\hat{\boldsymbol{\eta}} + G_0(s)\hat{\mathbf{z}}]\,\mathrm{d}s\end{aligned} \tag{7}$$

and

$$\mathbf{T}_D = -\frac{8\mu}{\pi}\int_{-\pi}^{\pi}\int_0^1 \left\{(\cos\phi\,\hat{\boldsymbol{\eta}} + \sin\phi\,\hat{\mathbf{z}}) \times [e(\alpha,\phi)\hat{\boldsymbol{\xi}} + f(\alpha,\phi)\hat{\boldsymbol{\eta}} + g(\alpha,\phi)\hat{\mathbf{z}}]\right\}\alpha^2\,\mathrm{d}\alpha\,\mathrm{d}\phi$$
$$= 32\mu\int_0^1 \left\{[F_{1s}(s) - G_{1c}(s)]\hat{\boldsymbol{\xi}} - E_{1s}(s)\hat{\boldsymbol{\eta}} + E_{1c}(s)\hat{\mathbf{z}}\right\} s\,\mathrm{d}s, \tag{8}$$

after substitution of (5) and (6).

For the velocity field, the stokeslet distribution is written as

$$\frac{1}{\pi^2}[(e\cos\beta + f\sin\beta)\hat{\mathbf{x}} + (f\cos\beta - e\sin\beta)\hat{\mathbf{y}} + g\hat{\mathbf{z}}],$$

in order to distinguish components normal and parallel to the wall. Thus

$$\mathbf{v}(x,y,z) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_0^1 \left\{[e(\alpha,\phi)\cos\beta + f(\alpha,\phi)\sin\beta]\left[\frac{\hat{\mathbf{x}}}{r} + \frac{x-\alpha\sin\beta\cos\phi}{r^3}\mathbf{r} - \mathbf{V}\right]\right.$$
$$+[-e(\alpha,\phi)\sin\beta + f(\alpha,\phi)\cos\beta]\left[\frac{\hat{\mathbf{y}}}{r} + \frac{y-\alpha\cos\beta\cos\phi}{r^3}\mathbf{r} - \mathbf{U}\right]$$
$$\left. + g(\alpha,\phi)\left[\frac{\hat{\mathbf{z}}}{r} + \frac{z-\alpha\sin\phi}{r^3}\mathbf{r} - \mathbf{U}^*\right]\right\}\alpha\,\mathrm{d}\alpha\,\mathrm{d}\phi$$
$$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_0^1 \left\{[e(\alpha,\phi)\hat{\boldsymbol{\xi}} + f(\alpha,\phi)\hat{\boldsymbol{\eta}} + g(\alpha,\phi)\hat{\mathbf{z}}]\frac{1}{r}\right.$$
$$+[e(\alpha,\phi)\xi + f(\alpha,\phi)(\eta - \alpha\cos\phi) + g(\alpha,\phi)(z - \alpha\sin\phi)]\frac{\mathbf{r}}{r^3}$$
$$-[e(\alpha,\phi)\cos\beta + f(\alpha,\phi)\sin\beta]\mathbf{V} - [-e(\alpha,\phi)\sin\beta + f(\alpha,\phi)\cos\beta]\mathbf{U}$$
$$\left. - g(\alpha,\phi)\mathbf{U}^*\right\}\alpha\,\mathrm{d}\alpha\,\mathrm{d}\phi, \tag{9}$$

where

$$\mathbf{r} = \xi\hat{\boldsymbol{\xi}} + (\eta - \alpha\cos\phi)\hat{\boldsymbol{\eta}} + (z - \alpha\sin\phi)\hat{\mathbf{z}} \tag{10}$$

and the reflected velocity fields $\mathbf{V}, \mathbf{U}$ and $\mathbf{U}^*$ are given, from [16] and as in [9], in terms of image singularities at

$$\mathbf{R} = (2D - x - \alpha\sin\beta\cos\phi)\hat{\mathbf{x}} + (y - \alpha\sin\beta\cos\phi)\hat{\mathbf{y}} + (z - \alpha\sin\phi)\hat{\mathbf{z}}, \tag{11}$$

by

$$V_x = \frac{2}{R} + \frac{4}{R^3}(D-x)(D - \alpha\sin\beta\cos\phi)$$
$$-\frac{(y-\alpha\sin\beta\cos\phi)^2 + (z-\alpha\sin\phi)^2}{R^3}\left[1 + \frac{6}{R^2}(D-x)(D-\alpha\sin\beta\cos\phi)\right],$$
$$(U_y, U_z^*) = (1,1)\left[\frac{1}{R} + \frac{2}{R^3}(D-x)(D - \alpha\sin\beta\cos\phi)\right]$$
$$+\frac{[(y-\alpha\sin\beta\cos\phi)^2, (z-\alpha\sin\phi)^2]}{R^3}\left[1 - \frac{6}{R^2}(D-x)(D-\alpha\sin\beta\cos\phi)\right],$$
$$(U_z, U_y^*) = (1,1)\frac{(y-\alpha\sin\beta\cos\phi)(z-\alpha\sin\phi)}{R^3}\left[1 - \frac{6}{R^2}(D-x)(D-\alpha\sin\beta\cos\phi)\right],$$

$$
\begin{aligned}
(U_x, U_x^*) &= \frac{(y - \alpha\sin\beta\cos\phi, z - \alpha\sin\phi)}{R^3}\Big[x - \alpha\sin\beta\cos\phi \\
&\quad + \frac{6}{R^2}(D - x)(D - \alpha\sin\beta\cos\phi)(2D - x - \alpha\sin\beta\cos\phi)\Big], \\
(V_y, V_x) &= \frac{(y - \alpha\sin\beta\cos\phi, z - \alpha\sin\phi)}{R^3}\Big[x - \alpha\sin\beta\cos\phi \\
&\quad - \frac{6}{R^2}(D - x)(D - \alpha\sin\beta\cos\phi)(2D - x - \alpha\sin\beta\cos\phi)\Big].
\end{aligned} \tag{12}
$$

Now, since $\xi = 0, (\eta, z) = \rho(\cos\theta, \sin\theta)(0 \leqslant \rho \leqslant 1, -\pi < \theta \leqslant \pi)$, i.e $(x, y) = \rho\cos\theta(\sin\beta, \cos\beta)$ from (4), on the disk, the disk condition (2) yields the vector integral equation

$$
\begin{aligned}
&\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_0^1 \Big\{[e(\alpha,\phi)\hat{\boldsymbol{\xi}} + f(\alpha,\phi)\hat{\boldsymbol{\eta}} + g(\alpha,\phi)\hat{\mathbf{z}}]\frac{1}{r} \\
&\quad + [f(\alpha,\phi)(\rho\cos\theta - \alpha\cos\phi) + g(\alpha,\phi)(\rho\sin\theta - \alpha\sin\phi)]\frac{\mathbf{r}}{r^3}\Big\}\,\alpha\,\mathrm{d}\alpha\,\mathrm{d}\phi \\
&= (U_0\cos\beta - V_0\sin\beta)\hat{\boldsymbol{\xi}} + (U_0\sin\beta + V_0\cos\beta)\hat{\boldsymbol{\eta}} + W_0\hat{\mathbf{z}} \\
&\quad + \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_0^1 \{[e(\alpha,\phi)\cos\beta + f(\alpha,\phi)\sin\beta]\mathbf{V} \\
&\quad + [-e(\alpha,\phi)\sin\beta + f(\alpha,\phi)\cos\beta]\mathbf{U} + g(\alpha,\phi)\mathbf{U}^*\}\alpha\,\mathrm{d}\alpha\,\mathrm{d}\phi,
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
\mathbf{r} &= (\rho\cos\theta - \alpha\cos\phi)\hat{\boldsymbol{\eta}} + (\rho\sin\theta - \alpha\sin\phi)\hat{\mathbf{z}}, \\
r^2 &= \rho^2 + \alpha^2 - 2\rho\alpha\cos(\theta - \phi).
\end{aligned}
$$

3. Solution to $O(D^{-1})$

The principal effect of the wall on the disk, at large enough separation, D, can be determined by first noting from (12) that

$$
\begin{aligned}
&V_x = \frac{3}{2D} + O\left(\frac{1}{D^2}\right), \qquad U_y, U_z^* = \frac{3}{4D} + O\left(\frac{1}{D^2}\right), \\
&U_x, U_x^*, V_y, V_z = O\left(\frac{1}{D^2}\right), \qquad U_z, U_y^* = O\left(\frac{1}{D^3}\right),
\end{aligned} \tag{14}
$$

which shows that the leading terms on the right-hand side of (13) due to the reflected velocities are

$$
\begin{aligned}
&\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_0^1 \Big\{[e(\alpha,\phi)\cos\beta + f(\alpha,\phi)\sin\beta]\frac{3}{2D}\hat{\mathbf{x}} \\
&\quad + [-e(\alpha,\phi)\sin\beta + f(\alpha,\phi)\cos\beta]\hat{\mathbf{y}} + g(\alpha,\phi)\frac{3}{4D}\hat{\mathbf{z}}\Big\}\,\alpha\,\mathrm{d}\alpha\,\mathrm{d}\phi \\
&= \frac{3}{2\pi D}\int_0^1 \{E_0(s)(2\cos^2\beta + \sin^2\beta)\hat{\boldsymbol{\xi}} + F_0(s)\sin\beta\cos\beta\,\hat{\boldsymbol{\xi}} \\
&\quad + E_0(s)\sin\beta\cos\beta\,\hat{\boldsymbol{\eta}} + F_0(s)(2\sin^2\beta + \cos^2\beta)\hat{\boldsymbol{\eta}} + G_0(s)\hat{\mathbf{z}}\}s\,\mathrm{d}s,
\end{aligned}
$$

after using (4), (5) and (6) to obtain this constant vector. Thus, from (13), the total broadside velocity of the disk relative to the fluid is effectively

$$U_0 \cos\beta - V_0 \sin\beta + \frac{3}{4\pi D}\int_0^1 \{E_0(s)(3+\cos 2\beta) + F_0(s)\sin 2\beta\}\,\mathrm{d}s,$$

while the total edgewise velocity is effectively

$$(U_0 \sin\beta + V_0\cos\beta)\hat{\boldsymbol{\eta}} + W_0\hat{\mathbf{z}} + \frac{3}{4\pi D}\hat{\boldsymbol{\eta}}\int_0^1 \{E_0(s)\sin 2\beta + F_0(s)(3-\cos 2\beta)\}\,\mathrm{d}s + \frac{3}{2\pi D}\hat{\mathbf{z}}\int_0^1 G_0(s)\,\mathrm{d}s.$$

Comparison with known results for a disk moving broadside or edgewise [17] in unbounded fluid readily shows now that the density functions on the left-hand side of (13) are, after use of (6), such that

$$E_0(s) \equiv \tilde{E}_0 = U_0\cos\beta - V_0\sin\beta + \frac{3}{4\pi D}\{\tilde{E}_0(3+\cos 2\beta) + \tilde{F}_0\sin 2\beta\} \tag{15}$$

and

$$\frac{3}{2}[F_0(s)\hat{\boldsymbol{\eta}} + G_0(s)\hat{\mathbf{z}}] \equiv \frac{3}{2}[\tilde{F}_0\hat{\boldsymbol{\eta}} + \tilde{G}_0\hat{\mathbf{z}}] = (U_0\sin\beta + V_0\cos\beta)\hat{\boldsymbol{\eta}} + W_0\hat{\mathbf{z}} + \frac{3}{4\pi D}\hat{\boldsymbol{\eta}}\{\tilde{E}_0\sin 2\beta + \tilde{F}_0(3-\cos 2\beta)\} + \frac{3\tilde{G}_0}{2\pi D}\hat{\mathbf{z}}. \tag{16}$$

Thus

$$\begin{aligned}
\tilde{E}_0 &= \Delta^{-1}\left[U_0\cos\beta\left(1-\frac{1}{\pi D}\right) - V_0\sin\beta\left(1-\frac{2}{\pi D}\right)\right],\\
\tilde{F}_0 &= \Delta^{-1}\left[U_0\sin\beta\left(\frac{2}{3}-\frac{1}{\pi D}\right) + V_0\cos\beta\left(\frac{2}{3}-\frac{2}{\pi D}\right)\right],\\
\tilde{G}_0 &= \frac{2}{3}W_0\left(1-\frac{1}{\pi D}\right)^{-1},
\end{aligned} \tag{17}$$

where

$$\Delta = 1 - \frac{1}{4\pi D}(15+\cos 2\beta) + \frac{3}{\pi^2 D^2}.$$

Further details of the above calculation will become apparent in the subsequent text. The drag, $\mathbf{F}_D$ is determined, to this order, by substitution of (17) in (7), while the torque, $\mathbf{T}_D$, given by (8), vanishes at this order in D^{-1}. The special cases are:

(i) $\beta = 0$ (axisymmetric geometry) in which

$$\mathbf{F}_D = -16\mu\left\{\frac{U_0\hat{\mathbf{x}}}{1-3/(\pi D)} + \frac{\frac{2}{3}V_0\hat{\mathbf{y}}}{1-1/(\pi D)} + \frac{\frac{2}{3}W_0\hat{\mathbf{z}}}{1-1/(\pi D)}\right\}, \tag{18}$$

due to broadside motion towards the wall and edgewise motion in two directions parallel to the wall, and

– (ii) $\beta = \pi/2$ in which

$$\mathbf{F}_D = -16\mu\left\{\frac{\frac{2}{3}U_0\hat{\mathbf{x}}}{1-2/(\pi D)} + \frac{V_0\hat{\mathbf{y}}}{1-3/(2\pi D)} + \frac{\frac{2}{3}W_0\hat{\mathbf{z}}}{1-1/(\pi D)}\right\}, \tag{19}$$

due respectively to edgewise motion towards the wall and broadside and edgewise motion parallel to the wall. These results are in agreement with [15].

4. Solution to $O(D^{-3})$

At this accuracy, the reflected velocities do not appear to be constant over the disk and so the wall effects can no longer be regarded as equivalent to those of an unbounded anisotropic medium. Then, as in [9], the full structure of the solution of the vector integral Eq. (13) must be considered. It was shown in [9] that the identity [18]

$$\begin{aligned}&[\rho^2+\alpha^2-2\rho\alpha\cos(\theta-\phi)+z^2]^{-1/2}\\ &\quad=\sum_{m=0}^{\infty}\epsilon_m\cos m(\theta-\phi)\int_0^\infty e^{-k|z|}J_m(k\rho)J_m(k\alpha)\,\mathrm{d}k,\end{aligned} \tag{20}$$

used below with $z=0$, implies that

$$\begin{aligned}&\frac{\rho^2 {}^{\cos}_{\sin} 2\theta + \alpha^2 {}^{\cos}_{\sin} 2\phi - 2\rho\alpha {}^{\cos}_{\sin}(\theta+\phi)}{[\rho^2+\alpha^2-2\rho\alpha\cos(\theta-\phi)]^{3/2}}\\ &\quad=\sum_{m=0}^{\infty}\frac{\epsilon_m}{2}\int_0^\infty\left\{J_{m+2}(k\rho)\,{}^{\cos}_{\sin}[(m+2)\theta-m\phi]\pm J_{m-2}(k\rho)\,{}^{\cos}_{\sin}[(m-2)\theta-m\phi]\right\}\\ &\quad\times J_m(k\alpha)\,\mathrm{d}k.\end{aligned} \tag{21}$$

Before applying these identities to the left-hand side of (13), it is convenient [9] now to note, from (6), that

$$\int_0^1 e_{nc}(\alpha)J_n(k\alpha)\alpha\,\mathrm{d}\alpha = \int_0^1 E_{nc}(s)h_{n-1}(ks)\,\mathrm{d}s \quad (n\geqslant 0), \tag{22}$$

where

$$h_n(ks) = \frac{k}{s^n}\int_0^s \frac{\rho^{n+1}J_n(k\rho)}{(s^2-\rho^2)^{1/2}}\,\mathrm{d}\rho = \left[\frac{\pi ks}{2}\right]^{1/2} J_{n+1/2}(ks) \quad (n\geqslant -1), \tag{23}$$

whence, in particular,

$$h_{-1}=\cos ks,\quad h_0=\sin ks,\quad h_1=\frac{\sin ks}{ks}-\cos ks. \tag{24}$$

Meanwhile, the reflected velocities on the right-hand side of (13) are, from (12), such that

$$\begin{aligned}V_x &= \frac{3}{2D}+\frac{3\sin\beta}{4D^2}(\alpha\cos\phi+\rho\cos\theta)-\frac{5}{8D^3}(\rho^2+\alpha^2)\\ &+\frac{7\sin^2\beta}{16D^3}(\rho^2+\alpha^2+\rho^2\cos 2\theta+\alpha^2\cos 2\phi)\\ &+\frac{\rho\alpha}{4D^3}[5\cos(\theta-\phi)-\sin^2\beta\cos\theta\cos\phi]+O\left(\frac{1}{D^4}\right),\end{aligned}$$

$$
\begin{aligned}
(U_y, U_z^*) &= (1,1)\left\{\frac{3}{4D} + \frac{3\sin\beta}{8D^2}(\alpha\cos\phi + \rho\cos\theta) - \frac{5}{32D^3}(\rho^2+\alpha^2)\right.\\
&\quad + \frac{9\sin^2\beta}{64D^3}(\rho^2+\alpha^2+\rho^2\cos 2\theta + \alpha^2\cos 2\phi)\\
&\quad \left. + \frac{\rho\alpha}{16D^3}[5\cos(\theta-\phi) + 3\sin^2\beta\cos\theta\cos\phi]\right\}\\
&\quad - \frac{1}{32D^3}\Big[(\cos^2\beta, 1)(\rho^2+\alpha^2) + (\cos^2\beta,-1)(\rho^2\cos 2\theta + \alpha^2\cos 2\phi)\\
&\quad - 4\rho\alpha(\cos^2\beta\cos\theta\cos\phi, \sin\theta\sin\phi)\Big] + O\left(\frac{1}{D^4}\right),\\
(V_y, U_x) &= (1,-1)\frac{3\cos\beta}{8D^2}\left[\alpha\cos\phi - \rho\cos\theta - \frac{\sin\beta}{2D}(\rho^2-\alpha^2+\rho^2\cos 2\theta - \alpha^2\cos 2\phi)\right]\\
&\quad + (1,1)\frac{\sin 2\beta}{32D^3}[\rho^2+\alpha^2+\rho^2\cos 2\theta + \alpha^2\cos 2\phi - 4\rho\alpha\cos\theta\cos\phi] + O\left(\frac{1}{D^4}\right),\\
(V_z, U_x^*) &= (1,-1)\frac{3}{8D^2}\left[\alpha\sin\phi - \rho\sin\theta + \frac{\sin\beta}{2D}(\alpha^2\sin 2\phi - \rho^2\sin 2\theta)\right.\\
&\quad \left. - \frac{\sin\beta}{D}\rho\alpha\sin(\theta-\phi)\right] + (1,1)\frac{\sin\beta}{16D^3}[\alpha^2\sin 2\phi\\
&\quad + \rho^2\sin 2\theta - 2\rho\alpha\sin(\theta+\phi)] + O\left(\frac{1}{D^4}\right),\\
(U_z, U_y^*) &= (-1,-1)\frac{\cos\beta}{32D^3}[\alpha^2\sin 2\phi + \rho^2\sin 2\theta - 2\rho\alpha\sin(\theta+\phi)] + O\left(\frac{1}{D^4}\right).
\end{aligned}
\tag{25}
$$

Inspection of the various expressions in (25) shows that all terms that depend on ϕ are no larger than $O(D^{-2})$, which property is shared by the density functions, according to the previous section. Consequently, since the right-hand side of (13) is a mean value over ϕ and is to be evaluated to $O(D^{-3})$ only, terms with ϕ-dependence can be discarded from (25) in evaluating the torque and drag on the disk to $O(D^{-3})$.

Thus, with the aid of (5), (20), (22) and (25), the $\hat{\xi}$ component of (13) yields

$$
\begin{aligned}
&\frac{2}{\pi}\int_0^1 \sum_0^\infty \epsilon_n[E_{nc}(s)\cos n\theta + E_{ns}(s)\sin n\theta]\int_0^\infty J_n(k\rho)h_{n-1}(ks)\,\mathrm{d}k\,\mathrm{d}s\\
&= U_0\cos\beta - V_0\sin\beta + \frac{2}{\pi}\int_0^1 [E_0(s)\cos\beta + F_0(s)\sin\beta]\\
&\times\left\{\cos\beta\left[\frac{3}{2D} + \frac{3\sin\beta}{4D^2}\rho\cos\theta + \frac{7\sin^2\beta - 10}{16D^3}(\rho^2+2s^2) + \frac{7\sin^2\beta}{16D^3}\rho^2\cos 2\theta\right]\right.\\
&\left. + \sin\beta\left[\frac{3\cos\beta}{8D^2}\rho\cos\theta + \frac{\sin 2\beta}{16D^3}(\rho^2+\rho^2\cos 2\theta - 4s^2)\right]\right\}\mathrm{d}s\\
&+ \frac{2}{\pi}\int_0^1 [-E_0(s)\sin\beta + F_0(s)\cos\beta]\left\{\cos\beta\left[\frac{3\cos\beta}{8D^2}\rho\cos\theta\right.\right.\\
&\left. + \frac{\sin 2\beta}{16D^3}(2\rho^2 + 2\rho^2\cos 2\theta - 2s^2)\right]\\
&- \sin\beta\left[\frac{3}{4D} + \frac{3\sin\beta}{8D^2}\rho\cos\theta + \frac{9\sin^2\beta - 10 - 2\cos^2\beta}{64D^3}(\rho^2+2s^2)\right.
\end{aligned}
$$

$$+\frac{9\sin^2\beta-2\cos^2\beta}{64D^3}\rho^2\cos 2\theta\Bigg]\Bigg\}\,\mathrm{d}s+\frac{2}{\pi}\int_0^1 G_0(s)\,\mathrm{d}s\left[\frac{3\cos\beta}{8D^2}\rho\sin\theta\right.$$

$$\left.+\frac{9\sin 2\beta}{64D^3}\rho^2\sin 2\theta\right]. \tag{26}$$

Similarly, with the additional use of (21), the $\hat{\eta}$ and $\hat{\mathbf{z}}$ components of (13) yield

$$\frac{3}{\pi}\int_0^1\sum_0^\infty \epsilon_n[F_{nc}(s)\cos n\theta+F_{ns}(s)\sin n\theta]\int_0^\infty J_n(k\rho)h_{n-1}(ks)\,\mathrm{d}k\,\mathrm{d}s$$

$$+\frac{1}{2\pi}\int_0^1\sum_0^\infty \epsilon_n\{[F_{nc}(s)-G_{ns}(s)]\cos(n+2)\theta$$

$$+[F_{ns}(s)+G_{nc}(s)]\sin(n+2)\theta\}\int_0^\infty J_{n+2}(k\rho)h_{n-1}(ks)\,\mathrm{d}k\,\mathrm{d}s$$

$$+\frac{1}{2\pi}\int_0^1\sum_0^\infty \epsilon_n\{[F_{nc}(s)+G_{ns}(s)]\cos(n-2)\theta$$

$$+[F_{ns}(s)-G_{nc}(s)]\sin(n-2)\theta\}\int_0^\infty J_{n-2}(k\rho)h_{n-1}(ks)\,\mathrm{d}k\,\mathrm{d}s$$

$$=U_0\sin\beta+V_0\cos\beta+\frac{2}{\pi}\int_0^1[E_0(s)\cos\beta+F_0(s)\sin\beta]$$

$$\left\{\sin\beta\left[\frac{3}{2D}+\frac{3\sin\beta}{4D^2}\rho\cos\theta+\frac{7\sin^2\beta-10}{16D^3}(\rho^2+2s^2)+\frac{7\sin^2\beta}{16D^3}\rho^2\cos 2\theta\right]\right.$$

$$\left.-\cos\beta\left[\frac{3\cos\beta}{8D^2}\rho\cos\theta+\frac{\sin 2\beta}{16D^3}(\rho^2+\rho^2\cos 2\theta-4s^2)\right]\right\}\mathrm{d}s$$

$$+\frac{2}{\pi}\int_0^1[-E_0(s)\sin\beta+F_0(s)\cos\beta]\left\{\sin\beta\left[\frac{3\cos\beta}{8D^2}\rho\cos\theta\right.\right.$$

$$\left.+\frac{\sin 2\beta}{16D^3}(2\rho^2+2\rho^2\cos 2\theta-2s^2)\right]$$

$$+\cos\beta\left[\frac{3}{4D}+\frac{3\sin\beta}{8D^2}\rho\cos\theta+\frac{9\sin^2\beta-10-2\cos^2\beta}{64D^3}(\rho^2+2s^2)\right.$$

$$\left.\left.+\frac{9\sin^2\beta-2\cos^2\beta}{64D^3}\rho^2\cos 2\theta\right]\right\}\mathrm{d}s$$

$$+\frac{2}{\pi}\int_0^1 G_0(s)\,\mathrm{d}s\left[\frac{3\sin\beta}{8D^2}\rho\sin\theta+\frac{8\sin^2\beta-\cos^2\beta}{32D^3}\rho^2\sin 2\theta\right] \tag{27}$$

and

$$\frac{3}{\pi}\int_0^1\sum_0^\infty \epsilon_n[G_{nc}(s)\cos n\theta+G_{ns}(s)\sin n\theta]\int_0^\infty J_n(k\rho)h_{n-1}(ks)\,\mathrm{d}k\,\mathrm{d}s$$

$$-\frac{1}{2\pi}\int_0^1\sum_0^\infty \epsilon_n\{[F_{ns}(s)+G_{nc}(s)]\cos(n+2)\theta$$

$$-[F_{nc}(s)-G_{ns}(s)]\sin(n+2)\theta\}\int_0^\infty J_{n+2}(k\rho)h_{n-1}(ks)\,\mathrm{d}k\,\mathrm{d}s$$

$$+\frac{1}{2\pi}\int_0^1\sum_0^\infty \epsilon_n\{[F_{ns}(s)-G_{nc}(s)]\cos(n-2)\theta$$

$$-[F_{nc}(s)+G_{ns}(s)]\sin(n-2)\theta\}\int_0^\infty J_{n-2}(k\rho)h_{n-1}(ks)\,\mathrm{d}k\,\mathrm{d}s$$

$$=W_0-\frac{2}{\pi}\int_0^1[E_0(s)\cos\beta+F_0(s)\sin\beta]\,\mathrm{d}s\left[\frac{3}{8D^2}\rho\sin\theta+\frac{\sin\beta}{8D^3}\rho^2\sin 2\theta\right]$$

$$-\frac{2}{\pi}\int_0^1[-E_0(s)\sin\beta+F_0(s)\cos\beta]\,\mathrm{d}s\frac{\cos\beta}{32D^3}\rho^2\sin 2\theta+\frac{2}{\pi}\int_0^1 G_0(s)$$

$$\times\left[\frac{3}{4D}+\frac{3\sin\beta}{8D^2}\rho\cos\theta+\frac{9\sin^2\beta-12}{64D^3}(\rho^2+2s^2)+\frac{9\sin^2\beta+2}{64D^3}\rho^2\cos 2\theta\right]\mathrm{d}s. \tag{28}$$

Since the right-hand sides of Eqs. (26–28) contain only the zeroth, first and second Fourier modes in θ, consider now the fifteen equations obtained from these components of each equation. The resulting Abel integral equations are readily solved by noting that the identity, obtained from (23),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{\rho^{n+1}J_n(k\rho)}{(t^2-\rho^2)^{1/2}}\,\mathrm{d}\rho=t^n h_{n-1}(kt)\quad(n\geqslant 0) \tag{29}$$

enables the remaining Bessel functions to be replaced by h_{-1}, h_0 or h_1 by applying the operator

$$\frac{1}{t^n}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{\rho^{n+1}}{(t^2-\rho^2)^{1/2}}\,\mathrm{d}\rho\quad(0\leqslant t\leqslant 1)$$

to the fifteen equations with $n=0,1$ or 2. The solution of each integral equation is completed by using

$$\frac{2}{\pi}\int_0^\infty h_{n-1}(kt)h_{n-1}(ks)\,\mathrm{d}k=\delta(s-t)\quad(n\geqslant 0),$$

$$\frac{2}{\pi}\int_0^\infty h_{n-1}(kt)h_{n+1}(ks)\,\mathrm{d}k=(2n+1)\frac{t^n}{s^{n+1}}H(s-t)-\delta(s-t)\quad(n\geqslant 0), \tag{30}$$

obtained, as in [9], by substitution of (23) and suitable manipulation of the identity

$$\int_0^\infty J_\nu(ks)J_{\nu-1}(kt)\,\mathrm{d}k=\frac{t^{\nu-1}}{s^\nu}H(s-t),$$

[18, Section 6.512], with $\delta(x)$ and $H(x)$ denoting the Dirac delta and Heaviside unit functions, respectively. Thus (26–28) yield

$$E_0(t)=U_0\cos\beta-V_0\sin\beta+\frac{2}{\pi}\int_0^1[E_0(s)\cos\beta+F_0(s)\sin\beta]$$

$$\times\left\{\cos\beta\left[\frac{3}{2D}+\frac{7\sin^2\beta-10}{8D^3}(t^2+s^2)\right]+\sin\beta\frac{\sin 2\beta}{8D^3}(t^2-2s^2)\right\}\mathrm{d}s$$

$$+\frac{2}{\pi}\int_0^1[-E_0(s)\sin\beta+F_0(s)\cos\beta]\left\{\cos\beta\frac{\sin 2\beta}{8D^3}(2t^2-s^2)\right.$$
$$\left.-\sin\beta\left[\frac{3}{4D}+\frac{9\sin^2\beta-10-2\cos^2\beta}{32D^3}(t^2+s^2)\right]\right\}\mathrm{d}s, \tag{31}$$

$$\frac{3}{2}F_0(t)+\frac{1}{\pi}\int_0^1[F_{2c}(s)+G_{2s}(s)]\int_0^\infty h_{-1}(kt)h_1(ks)\,\mathrm{d}k\,\mathrm{d}s$$
$$=U_0\sin\beta+V_0\cos\beta+\frac{2}{\pi}\int_0^1[E_0(s)\cos\beta+F_0(s)\sin\beta]$$
$$\times\left\{\sin\beta\left[\frac{3}{2D}+\frac{7\sin^2\beta-10}{8D^3}(t^2+s^2)\right]-\cos\beta\frac{\sin 2\beta}{8D^3}(t^2-2s^2)\right\}\mathrm{d}s$$
$$+\frac{2}{\pi}\int_0^1[-E_0(s)\sin\beta+F_0(s)\cos\beta]\left\{\sin\beta\frac{\sin 2\beta}{8D^3}(2t^2-s^2)\right.$$
$$\left.+\cos\beta\left[\frac{3}{4D}+\frac{9\sin^2\beta-10-2\cos^2\beta}{32D^3}(t^2+s^2)\right]\right\}\mathrm{d}s, \tag{32}$$

$$\frac{3}{2}G_0(t)+\frac{1}{\pi}\int_0^1[F_{2s}(s)-G_{2c}(s)]\int_0^\infty h_{-1}(kt)h_1(ks)\,\mathrm{d}k\,\mathrm{d}s$$
$$=W_0+\frac{2}{\pi}\int_0^1 G_0(s)\left[\frac{3}{4D}+\frac{9\sin^2\beta-12}{32D^3}(t^2+s^2)\right]\mathrm{d}s, \tag{33}$$

$$2E_{1c}(t)=\frac{2}{\pi}\int_0^1[E_0(s)\cos\beta+F_0(s)\sin\beta]\,\mathrm{d}s\frac{9\sin 2\beta}{8D^2}t$$
$$+\frac{2}{\pi}\int_0^1[-E_0(s)\sin\beta+F_0(s)\cos\beta]\,\mathrm{d}s\frac{3\cos 2\beta}{4D^2}t, \tag{34}$$

$$2E_{1s}(t)=\frac{2}{\pi}\int_0^1 G_0(s)\,\mathrm{d}s\frac{3\cos\beta}{4D^2}t, \tag{35}$$

$$3F_{1c}(t)-\frac{1}{2}[F_{1c}(t)+G_{1s}(t)]=\frac{2}{\pi}\int_0^1[E_0(s)\cos\beta+F_0(s)\sin\beta]\,\mathrm{d}s$$
$$\times\frac{3}{4D^2}(2\sin^2\beta-\cos^2\beta)t+\frac{2}{\pi}\int_0^1[-E_0(s)\sin\beta+F_0(s)\cos\beta]\,\mathrm{d}s\frac{3\sin 2\beta}{4D^2}t, \tag{36}$$

$$3G_{1c}(t)-\frac{1}{2}[F_{1s}(t)-G_{1c}(t)]=\frac{2}{\pi}\int_0^1 G_0(s)\,\mathrm{d}s\frac{3\sin\beta}{4D^2}t, \tag{37}$$

$$3F_{1s}(t)+\frac{1}{2}[F_{1s}(t)-G_{1c}(t)]=\frac{2}{\pi}\int_0^1 G_0(s)\,\mathrm{d}s\frac{3\sin\beta}{4D^2}t, \tag{38}$$

$$3G_{1s}(t)-\frac{1}{2}[F_{1c}(t)+G_{1s}(t)]=\frac{2}{\pi}\int_0^1[E_0(s)\cos\beta+F_0(s)\sin\beta]\,\mathrm{d}s\frac{3}{4D^2}t. \tag{39}$$

$$2E_{2c}(t) = \frac{2}{\pi}\int_0^1 [E_0(s)\cos\beta + F_0(s)\sin\beta]\,\mathrm{d}s\frac{3t^2}{4D^3}\sin\beta\sin 2\beta$$
$$+\frac{2}{\pi}\int_0^1 [-E_0(s)\sin\beta + F_0(s)\cos\beta]\,\mathrm{d}s\frac{3t^2\sin\beta}{8D^3}(2\cos^2\beta - \sin^2\beta),$$

$$2E_{2s}(t) = \frac{2}{\pi}\int_0^1 G_0(s)\,\mathrm{d}s\frac{3t^2\sin 2\beta}{8D^3},$$

$$3F_{2c}(t) + \frac{1}{\pi}\int_0^1 F_0(s)\int_0^\infty h_1(kt)h_{-1}(ks)\,\mathrm{d}k\,\mathrm{d}s$$
$$= \frac{2}{\pi}\int_0^1 [E_0(s)\cos\beta + F_0(s)\sin\beta]\,\mathrm{d}s\frac{t^2\sin\beta}{6D^3}(7\sin^2\beta - 2\cos^2\beta)$$
$$+\frac{2}{\pi}\int_0^1 [-E_0(s)\sin\beta + F_0(s)\cos\beta]\,\mathrm{d}s\frac{t^2\cos\beta}{24D^3}(25\sin^2\beta - 2\cos^2\beta),$$

$$3F_{2s}(t)+\frac{1}{\pi}\int_0^1 G_0(s)\int_0^\infty h_1(kt)h_{-1}(ks)\,\mathrm{d}k\,\mathrm{d}s=\frac{2}{\pi}\int_0^1 G_0(s)\,\mathrm{d}s\frac{t^2}{12D^3}(8\sin^2\beta-\cos^2\beta),$$

$$3G_{2c}(t)-\frac{1}{\pi}\int_0^1 G_0(s)\int_0^\infty h_1(kt)h_{-1}(ks)\,\mathrm{d}k\,\mathrm{d}s=\frac{2}{\pi}\int_0^1 G_0(s)\,\mathrm{d}s\frac{t^2}{24D^3}(9\sin^2\beta+2),$$

$$3G_{2s}(t) + \frac{1}{\pi}\int_0^1 F_0(s)\int_0^\infty h_1(kt)h_{-1}(ks)\,\mathrm{d}k\,\mathrm{d}s$$
$$= -\frac{2}{\pi}\int_0^1 [E_0(s)\cos\beta + F_0(s)\sin\beta]\,\mathrm{d}s\frac{t^2\sin\beta}{3D^3}$$
$$-\frac{2}{\pi}\int_0^1 [-E_0(s)\sin\beta + F_0(s)\cos\beta]\,\mathrm{d}s\frac{t^2\cos\beta}{12D^3}.$$

Two important simplifications are now available because (30) implies that

$$\int_0^1 t^n\int_0^\infty h_{n-1}(kt)h_{n+1}(ks)\,\mathrm{d}k\,\mathrm{d}t = 0 \quad (0 < s < 1). \tag{40}$$

One has already been used in discarding all higher modes of the density functions. This is justified because, in this approximation, the first and second mode functions are proportional to s and s^2, respectively and hence, with s and t interchanged and $n = 1$ or 2 in (40), their contributions to the integral equations for the third and fourth modes are zero. Secondly, for the purposes of the drag formula (7), the second mode functions, by virtue of (40), are irrelevant in (32) and (33). Further, all that is needed from (36–39) for the evaluation of the torque formula (8) is that (37) and (38) imply that

$$F_{1s} - G_{1c} = 0. \tag{41}$$

Consequently, by substitution in (31–35) of the $O(D^{-1})$ solution given by (15–17), it follows that the drag and torque components in (7) and (8) are now given to $O(D^{-3})$ by

$$\int_0^1 E_0(t)\,\mathrm{d}t = \tilde{E}_0 - \frac{\cos\beta}{36\pi D^3}[U_0(5+\cos 2\beta) - V_0\sin 2\beta](7 + 3\cos 2\beta)$$
$$+\frac{\sin\beta}{288\pi D^3}[-U_0\sin 2\beta + V_0(5 - \cos 2\beta)](17 + 15\cos 2\beta), \tag{42}$$

$$\int_0^1 F_0(t)\,\mathrm{d}t = \tilde{F}_0 - \frac{\sin\beta}{18\pi D^3}[U_0(5+\cos 2\beta) - V_0\sin 2\beta](2+\cos 2\beta) - \frac{\cos\beta}{144\pi D^3}[-U_0\sin 2\beta + V_0(5-\cos 2\beta)](3+5\cos 2\beta), \tag{43}$$

$$\int_0^1 G_0(t)\,\mathrm{d}t = \tilde{G}_0 - \frac{W_0}{36\pi D^3}(5+3\cos 2\beta), \tag{44}$$

$$\int_0^1 E_{1c}(t)t\,\mathrm{d}t = \frac{1}{4\pi D^2}[\tilde{E}_0\sin\beta(1+\cos^2\beta) + \tilde{F}_0\cos\beta(1+\sin^2\beta)], \tag{45}$$

$$\int_0^1 E_{1s}(t)\,\mathrm{d}t = \frac{\cos\beta}{4\pi D^2}\tilde{G}_0, \tag{46}$$

with (41) implying that the torque about the disk's axis of symmetry is zero. The quantities $\tilde{E}_0, \tilde{F}_0$ and $\tilde{G}_0$ in (42–46) are given by (17), which is now to be evaluated to $O(D^{-3})$.

The special cases are:

(i) $\beta = 0$ (axisymmetric geometry) in which

$$\mathbf{F}_D = -16\mu\left\{\frac{U_0\hat{\mathbf{x}}}{1-3/(\pi D)} + \frac{\frac{2}{3}V_0\hat{\mathbf{y}}}{1-1/(\pi D)} + \frac{\frac{2}{3}W_0\hat{\mathbf{z}}}{1-1/(\pi D)} - \frac{5U_0\hat{\mathbf{x}} + \frac{2}{3}V_0\hat{\mathbf{y}} + \frac{2}{3}W_0\hat{\mathbf{z}}}{3\pi D^3}\right\}, \tag{47}$$

$$\mathbf{T}_D = \frac{16\mu}{3\pi D^2}\frac{V_0\hat{\mathbf{z}} - W_0\hat{\mathbf{y}}}{1-2/(\pi D)}, \tag{48}$$

due to broadside motion towards the wall and edgewise motion in two directions parallel to the wall, and

(ii) $\beta = \pi/2$ in which

$$\mathbf{F}_D = -16\mu\left\{\frac{\frac{2}{3}U_0\hat{\mathbf{x}}}{1-2/(\pi D)} + \frac{V_0\hat{\mathbf{y}}}{1-3/(2\pi D)} + \frac{\frac{2}{3}W_0\hat{\mathbf{z}}}{1-1/(\pi D)} - \frac{\frac{2}{3}U_0\hat{\mathbf{x}} + \frac{1}{8}V_0\hat{\mathbf{y}} + \frac{1}{6}W_0\hat{\mathbf{z}}}{3\pi D^3}\right\}, \tag{49}$$

$$\mathbf{T}_D = -\frac{8\mu}{\pi D^2}\frac{V_0\hat{\mathbf{z}}}{1-3/(2\pi D)}, \tag{50}$$

due respectively to edgewise motion towards the wall and broadside and edgewise motion parallel to the wall. The $\hat{\mathbf{x}}$ components of the results (47) and (49) are, respectively, in agreement with the asymptotic result given in [14] and the $O(D^{-5})$ formula derived in [9] for broadside and edgewise motion towards a wall. The edgewise motions in the two cases involve different alignments of the disk relative to the wall, but the corresponding forces differ only at order D^{-3}. The torque expression (48) shows that in edgewise motion parallel to a wall with axisymmetric geometry the front end of the disk tends to rotate away from the wall. Meanwhile, (50) shows that in broadside motion parallel to a wall the torque tends to cause a rotation that would reduce the speed of the half of the disk closer to the wall. No torque is generated when the disk moves edgewise parallel to the wall with its axis similarly oriented, because the disk then lies in a plane of symmetry of the motion.

In conclusion, the drag and torque formulas, given by substitution of (41–46) in (7) and (8), for a unit disk translating arbitrarily at distance D from a plane wall, are uniformly valid to order D^{-3}, provided that the quasi-static approximation remains sensible. Evidently, the method can be continued to greater accuracy and could be adapted to consider the interaction of two disks translating in unbounded fluid.

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